The Properties of a SIX-DIMENSIONAL, pseudo-Riemannian Manifold

# Abstract

Here we describe a six-dimensional, pseudo-Riemannian manifold with three imaginary dimensions of time and three real dimensions of space, we assign it some simple rules, and then describe the resulting properties. Any two temporal dimensions form an imaginary plane. Each spatial dimension is orthogonal to the imaginary plane and the extent of the spatial dimension is the dot product of the temporal coordinates plus an offset. We then discuss the properties that this manifold possesses when projected onto the spatial dimensions and given a single evolution parameter. Here we demonstrate that the spatial projection of this manifold expands with time and even accelerates as it evolves, independent of any energy, stress or momentum contained therein.

Some formal definitions are required for our discussion.

# Real and Imaginary

For the scope of this paper, real is that which can be measured directly: space, acceleration, and the square root of a positive area, for example, while imaginary is that which isn’t real and, thus, can only be inferred: time, velocity, and the square root of a negative area, for example.

# Time

Imagine a line. Give it a distinct origin. This is time.

# Squared Time

Time, by itself, is unremarkable, but if you have two dimensions of time in an imaginary plane, then some interesting properties emerge. A third set of coordinates – a dimension of squared time – exists as the dot product of the temporal coordinates plus an offset.

Because squared time is a secondary dimension produced from two primary dimensions, a chromatic index is used. For example, magenta squared time, , is the product of the red time coordinate, , and the blue time coordinate, . This can be expressed as:

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Where is the extent of the squared time dimension, and are constant and have a dimension of time, . is the angle between the temporal coordinate axes and the cosine of this angle, , is the derivative . Here we extend the concept of a dot product with an offset, and, because the geometric definition of the dot product operator takes the norm of the vectors, we take some license and employ the shorthand notation of instead. This indicates that the basis vector has an imaginary component which cannot be ignored by the norm operator. This fixes the geometric definition so that it yields the same result as the algebraic definition when operating on two imaginary basis vectors. This operator also accommodates an offset which, we will demonstrate, serves a useful geometric purpose.

Squared time has the additional property of being real, making it the only dimension that can be measured directly. From Eq. (1) we can see that the dimension of squared time expands quadratically as a function of time. That is, when both temporal coordinates advance at the same rate, all points will move away from each other with a constant acceleration. On this manifold, objects at rest accelerate.

# A Meter

Squared time and space are the same thing, but with different units. Since time is imaginary, a conversion to a real number is needed to make meaningful measurements.

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Where is the constant ratio of space to time. Dimensionally, this constant is a velocity and the basis vector for the velocity coordinate system is . is the extent of the spatial dimension in some real unit that we’ll generically refer to as a meter, is the conversion factor between space and squared time.

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Figure 1 - The relationship between red time, , blue time, , and magenta space, . The size of the spatial dimension is proportional to the dot product of the temporal coordinates. Space is always orthogonal to time.

To further illustrate the meaning of real and imaginary (for the scope of this paper), draw a flatlander on your desk and place a cube next to it. To the flatlander, the cube can only be imagined, it can’t be measured with any flatlander instrument. However, to someone who exists in three dimensions, the cube is real and can be measured easily with a ruler. Space, then, is a lower, one-dimensional projection (scalar) of a negative area in a higher dimension, time. Conversely, time is the square root of a negative area.

# A Second

Hold a ruler in your hand. Space on this manifold can be measured with a ruler, however, being pseudo-Reimann, we must also measure the imaginary part. Now, imagine a ruler. Is it bigger or smaller than the one in your hand? Is it twice as big, half as big? Exactly how do we objectively compare something real with something imaginary?

Velocity relates space to time, so we could use a unit of space, and a given velocity, to define a unit of time. But which velocity to use? All observers in all reference frames must agree to use the same velocity when constructing this inferred coordinate axis or they will never agree on the distance between spacetime events.

We can exploit the fact that all points on a differentiable manifold have a well-defined tangent velocity. A second, then, is the projection of a temporal length, , through the tangent velocity, , and onto the temporal dimension giving us a spatial length, .

When two temporal axes are aligned, they form a reference frame, and the time coordinates advance synchronously such that , , and . Eq. (2) then reduces to:

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From this, we obtain an expression for the tangent velocity on the surfaced of this manifold:

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And taking the next derivative, and expression for the acceleration of the manifold:

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Where is the constant acceleration of the manifold, is the constant of integration (that is, the tangent velocity at ), and is the tangent velocity at time, . Knowing the tangent velocity of the reference frame, we derive our formal definition of a second in any frame, .

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With this choice of a coordinate system, meters are constant, seconds are variable and the length of a second increases as a function of time. Dimensionally, these seconds are spatial and, thus, are interchangeable with meters in every practical way. Note that this coordinate system possesses neither time symmetry nor Lorentz invariance.

# Tangent Space

At first blush, the fact that all points at a given time share the same tangent velocity would seem to make relative motion impossible. However, we can decompose the tangent vectors into smaller pieces to see if there is some room for relativity. We can define the space using the Pythagorean theorem.

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Where is the spacetime distance, is the metric tensor and is the difference between a set of coordinates in each dimension. As the spatial and temporal dimensions are orthogonal in this manifold, the magenta spacetime interval, , simply expands to:

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The tangent velocity can be expressed as the derivative of the tangent space with respect to the reference temporal axis, :

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We can now see the outline of relativity in components that allow for any relative velocity, defined as , where , so long as the tangent velocity is preserved. Substituting Eq. (5) and into Eq. (7), and solving for , we get:

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From this, the angle between the temporal axes can be expressed as a function of the relative velocity,

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And from this we find the relationship between the reference time, , and some arbitrary frame’s time, :

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From this relationship we can see that relative motion is possible so long as the sum of the squares of the component vectors adds up to the square of the tangent velocity. On this manifold, the faster an arbitrary frame is moving, the slower it will appear when project onto a reference frame.

# Motion

From Eq. (4) we can derive an expression for motion on the surface of this manifold. We assume that, in the absence of force, an object will follow a geodesic. This can be expressed as:

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Where is the basis vector for , and is the change in velocity. This can be expanded with the chain rule:

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Where is the acceleration due to some force, and is the acceleration due to the change in the basis vectors. In the absence of any force, Eq. (10) gives us the local curvature equivalent of the manifold. Note that the local curvature of spacetime not flat.

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In non-relativistic domains, our equation for motion becomes:

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Eq. (11) gives us the equation of motion for the path of a mass, , as it travels along the manifold in the presences of a collection of forces, .

# Spiral Galaxies

We can replace the general terms of Eq. (11) with more specific terms and derive an equation for centripetal motion in the presence of a gravitational force field:

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Where, for a given radius, *r, M(r)* is the enclosed mass, *v* is the angular velocity and *G* is the gravitational constant. This formula suggests that the effects of quadratic expansion would be apparent in low acceleration environments where . Spiral galaxies provide such an environment.

A mass model of a spiral galaxy is required to compare the prediction of equation (12) against the observations. A very simple mass model consists of a bulge and a disk. The bulge mass is modelled after the de Vaucouleurs profile (de Vaucouleurs 1958):

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Where is the mass of the bulge, is 7.6695 and is the density at the scale radius, . The mass for an exponentially thin disk, , is calculated from (Freeman 1970) as:

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where is the central surface density, is the scale length of the disk.

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Figure 2– The predicted angular velocities of quadratically expanding space (red) and the observed velocities (gray) of a selection of spiral galaxies. All horizontal axes are radius in kpc, vertical axes are angular velocity in km s-1.

This mass model is fitted to the data from (Sofue et al. 1999), (Sofue et al. 2003), (Sofue 2016), (Garrido et al. 2005), (Noordermeer et al. 2007) and (Martinsson et al. 2013). A sample of the resulting rotational curves can be found in Figure 2. Note the flattening of the rotation curve that is characteristic of centripetal motion in quadratically expanding space in a domain where the acceleration of the expansion balances the acceleration due to gravity.

# Plane of Mass

The formula for motion in Eq. (12) defines a plane of possible masses given two parameters: the angular velocity and the radius of orbiting objects. This formula can be rearranged to predict the maximum mass possible in a circular orbit given the tangential velocity and radius.

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Figure 3 - The plane of mass. Radius is in km, velocity in km s-1 and the vertical axis is in M⊙. Solid line is the total mass allowed by the rules of motion in quadratically expanding space.

The radius that encloses the total mass of Figure 3 is:

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Substituting the Eq. (16) back into Eq. (15) yields the formula for the total mass given the velocity:

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Where the velocity is in units of , and the total mass, , is in units of solar mass, . A study of the relation between velocity and mass was conducted in (McGaugh 2012). Those results are displayed in Figure 4 and overlaid with Eq. (17). The reduced of 0.65 demonstrates that the Baryonic Tully-Fisher Relationship (BTFR) is characteristic of centripetal motion in quadratically expanding space.

Figure 4 - The relation between angular velocity and mass. Circles are the combined gas and stellar mass of the gas-rich galaxies and the solid line is the maximum mass allowed by the formula for centripetal motion.

# Three-Dimensional Metric Formula

Using the spacetime interval of Eq. (6), and the relationship between time and space from Eq. (5), the metric formula in terms of a measurable (albeit indirect) value, is:

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Where is the time of the observation, and is the arbitrary time of some event. However, there is a problem with this formula. We have no way to directly measure . We cannot place a pin in the manifold at and use a ruler to measure how much space there is to because the point, , exists only in the past and rulers can only measure in the present. The schematic of Figure 2 illustrates that the only values that can be measured directly with a ruler (spatial coordinates) are and . We need to relate the coordinate distance, , to the line element, before this metric can be useful for measuring the real part of the manifold.

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Figure 5 - The spacetime interval, , from to in quadratically expanding spacetime showing the line elements, and .

Using Eq. (2), the relation between and , as the manifold expands, is:

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Where is the spatial extent at and is the spatial extent at . Next, we will observe that the line element, , is the coordinate distance, , less one half of expansion (expressed as ):

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The metric formula for a spacetime distance using practical coordinates is then:

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# Six-Dimensional Metric Formula

We now consider the permutations of space and time. Two dimensions of time result in only one spatial dimension, so any discussion would be of limited value. Three dimensions of time result in three dimensions of space, one for each imaginary plane. This option agrees with the observed inverse-square law, so let us put a pin in it. If we had four dimensions of time, then we would observe six dimensions of space. We do not, so this option also shows little promise.

Having considered the permutations, we will focus the discussion on a manifold with three imaginary dimensions of time and three real dimensions of space. We are going to label them as red time, , green time, and blue time, due to the way they produce secondary dimensions. The product of green time and red time is yellow space, . The product of blue time and green time is cyan space, .

The additional line elements are:

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These line elements form the metric formula for six-dimensional spacetime:

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# Four-Dimensional Metric Formula

In the special case where all three temporal dimensions are aligned, forming an inertial reference frame in six dimensions, this formula can be simplified according to the conditions:

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In addition, the constants, and , can be combined into three-plane aggregates:

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With these assumptions and substitutions, the metric formula for a four-dimensional approximation of the six-dimensional manifold with a single evolution parameter, , is:

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Where is the spacetime distance between two events, and are the temporal coordinates of the two events, and , , and are the spatial distances between those two events.

# INitial Conditions

From Eq. (24), we find three initial conditions for quadratically expanding space (QES): the acceleration, , the initial tangent velocity, , and the age of the manifold at the time of observation, .

If we chose a line-of-sign path along a null geodesic (that is, a path having and ) then the spatial distance between two events can be found by solving Eq. (24) for :

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This formula can be further simplified by encoding the time coordinates as:

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Where , the redshift, is the change in a unit length (wavelength) from to . This redshift value is encoded in photons, which we assume to travel along the null geodesic, making it possible to use the photons from a known source (of luminosity) as proxies for distance markers.

Chart

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Figure 6 - Top: The distances to a selection of 482 SNe Ia supernovae (green) and the distance predicted by the quadratically expandig space (QES) metric formula (red) and, for comparison, the distance predicted by the FLRW metric formula (blue). Bottom: Errors for each of the data points from the predicted values in Gpc.

Using the combined data from (Conley et al. 2011), (Rodney et al. 2012), (Jones et al. 2013), (Rodney et al. 2015) and the parameters from (Rodney, et al. 2015) to normalize the sets, we can extract the initial conditions using a chi-square minimization algorithm. A of tells us that the SNe Ia provide, not only an excellent match to the model, but also a reasonable set of initial conditions with which we can continue our discussion. This is our fiducial manifold.

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Table 1– The initial conditions.

# Einstein Field Equations

The Einstein Field Equations describes additional properties of this spacetime, but before we can use those equations, we must solve them. The Einstein Field Equation states that:

Where is the Ricci Tensor, is the Ricci Scalar, is the metric tensor, is the gravitational constant and is the covariant expression of the Energy Momentum Tensor.

For the metric tensor, we take the derivative of metric formula in Eq. (24) with respect to time. For the derivation, we assume that the time of the observation (terminus of the geodesic), , is constant and is arbitrary and allowed to vary as time, .

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The Christoffel Symbols, , are collected in Table (2).

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Table 2 – The Christoffel Symbols for quadratically expanding space.

The Ricci Tensor, , is derived from the Riemann Tensor, . The Ricci Tensor is:

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And from this, we can derive the Ricci Scalar, :

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We model the contents of our fiduciary manifold with a perfect baryon fluid having four-velocity of and an Energy Momentum Tensor of:

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Where is the pressure of the fluid and is the baryonic density. The trace of this tensor, , is:

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Now we have enough information to calculate the covariant of the Energy Momentum Tensor, :

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With these terms in hand, the Einstein Field Equations for the four-dimensional approximation of a six-dimensional manifold are:

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With these equations and the initial conditions, we can extract the formulas and values for the baryon density, and the pressure, of our fiducial manifold at time, :

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Our fiduciary model contains roughly one proton per cubic meter giving it a total baryonic mass of Note that curvature, density, pressure, and mass are not random values, but rather immutable properties derived from the initial conditions.